Minimum Time-to-Climb Trajectories Using a Modified Sweep Method

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Abstract

In this paper, a new indirect method for solving optimal control problems, called *the modified sweep method*, is demonstrated numerically. The state-adjoint rate vector is expressed in terms of a set of basis vectors. These basis vectors give rise to *rate coordinates* that approximately identify the stable and unstable behavior. The state, adjoint, and approximate stable rate coordinates are integrated forward in time using the approximate unstable rate coordinate as an input whereas the approximate unstable rate coordinate is integrated in backward time using the approximate stable rate coordinate as an input. As a result, the forward and backward integrations do not amplify errors in the unknown boundary conditions. The method is illustrated for two example problems. The method possesses two important features. First, the forward and backward integration are always stable. Second, it is shown that the converged solution is indeed a solution of the original problem.

Introduction

The purpose of this research is to develop a new methodology for handling optimal control problems for which the dynamics evolve on two or more widely separated time-scales. It is known that solutions of two time-scale optimal control problems evolve in three distinct phases. The solution first decays rapidly to a slow solution, the slow solution then takes over during the middle segment, and finally the solution grows rapidly to meet the final boundary condition. Using the singular perturbation method, the zeroth-order solution for the first segment is obtained by solving an infinite horizon regulator problem [3]. As a first step towards handling the complete problem, an approach presented here for solving infinite horizon regulator problems.

The modified sweep method, based on the Computational Singular Perturbation (CSP) methodology [4, 5] is a way to solve infinite horizon regulator problems. In the modified sweep method, the state-adjoint rate vector is expressed as a linear combination of known basis vectors which give rise to *rate coordinates*. By choosing an appropriate basis, the directions of the stable and unstable dynamics are approximately identified. In both forward time the state and adjoint are integrated. However, in forward time the the stable rate coordinate is integrated using the *unstable* rate coordinate as an input whereas in backward time the unstable rate coordinate is integrated using the *stable* rate coordinate as an input. This new value for the unstable rate coordinate is used to obtain a more accurate solution on the ensuing forward integration. This process is repeated until convergence.

The modified sweep method is demonstrated on two problems. The first problem is the regulation of a mass connected to a spring with a nonlinear forcing function. This example demonstrates two important properties. First, the unstable behavior is suppressed. Second, with each iteration the solution improves. After several iterations the exact solution is found.

The second example is a minimum time problem for a supersonic aircraft. The solution in the left boundary layer starting at a point near Mach 1 is considered. For this problem, the modified sweep method is shown suppress the unstable behavior on each sweep. Furthermore, with each successive sweep the solution is shown to approach the exact solution.

Problem Formulation

Consider the problem of minimizing the cost

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function

$$J = \int_0^{t_f} \mathcal{L}\left[x(t), u(t)\right] dt \qquad (1)$$

subject to the dynamic constraint

$$\begin{aligned} \dot{x} &= f(x, y, u),\ x(0) = x_0 \\ \epsilon \dot{y} &= g(x, y, u),\ y(0) = y_0 \end{aligned} \qquad (2)$$

where $x \in \mathbb{R}^l$ and $y \in \mathbb{R}^k$, $l + k = n$, together form the *state*, $u \in \mathbb{R}^m$ is the *control*, and ϵ is a small parameter that identifies a time-scale separation in the dynamics. Eq. (2) is in the so called *standard form* because the dynamics on the slow and fast time-scale are identified explicitly by x and y, respectively. The parameter ϵ may arise naturally, or may be introduced artificially to identify the separation in the time-scales. The Hamiltonian is given by

$$H = \mathcal{L} + \lambda_x^T f + \lambda_y^T g \qquad (3)$$

where $\lambda_x \in \mathbb{R}^l$ and $\lambda_y \in \mathbb{R}^k$ are the *co-states* or *adjoints* corresponding to x and y, respectively. The adjoint equations are given by

$$\begin{aligned} \dot{\lambda}_x &= -\partial H / \partial x \\ \epsilon \dot{\lambda}_y &= -\partial H / \partial y \end{aligned} \qquad (4)$$

together with the boundary condition

$$\begin{aligned} \lambda_x(t_f) &= 0 \\ \lambda_y(t_f) &= 0 \end{aligned} . \qquad (5)$$

and the optimal control is given by

$$u^* = \arg \min_u H. \qquad (6)$$

It is assumed here that the optimal control can be found as an explicit function of the state and adjoint, i.e., $u^* = u^*(x, y, \lambda_x, \lambda_y)$. Substituting the optimal control into Eq. (2) and Eq. (4), we obtain

$$\begin{aligned} \dot{x} &= \partial H / \partial \lambda_x,\ x(0) = x_0 \\ \epsilon \dot{y} &= \partial H / \partial \lambda_y,\ y(0) = y_0 \\ \dot{\lambda}_x &= -\partial H / \partial x,\ \lambda_x(t_f) = 0 \\ \epsilon \dot{\lambda}_y &= -\partial H / \partial y,\ \lambda_y(t_f) = 0. \end{aligned} \qquad (7)$$

where H now refers to the Hamiltonian evaluated on the optimal control. The system of Eq. (7) forms an *Hamiltonian Two-Point Boundary Value Problem* (HTPBVP). It can be seen that in the standard form, the adjoints associated with the slow and fast variables are themselves slow and fast. Since the Hamiltonian is not an explicit function of time, it is constant along solutions to Eq. (7).

Time-Scale Structure

In the singular perturbation method, the solution to the HTPBVP of Eq. (7) is constructed as a sum of a left boundary layer solution, a slow solution, and a right boundary layer solution [3]. The state-adjoint vector is defined as

$$p = \begin{bmatrix} x \\ y \\ \lambda_x \\ \lambda_y \end{bmatrix} . \qquad (8)$$

The solution for $p(t)$ and the control $u(t)$ can then be written as [7]

$$p(t, \epsilon) = p_s(t, \epsilon) + p_f^l\left(\frac{t - t_0}{\epsilon}, \epsilon\right) + p_f^r\left(\frac{t_f - t}{\epsilon}, \epsilon\right) \qquad (9)$$

$$u(t, \epsilon) = u_s(t, \epsilon) + u_f^l\left(\frac{t - t_0}{\epsilon}, \epsilon\right) + u_f^r\left(\frac{t_f - t}{\epsilon}, \epsilon\right). \qquad (10)$$

The first terms on the right hand sides of both Eqs. (9) and (10) represent the slow solution; the second and third terms represent boundary-layer corrections to the slow solution near the initial and final times, respectively. Geometrically [3], there exists a slow invariant manifold in the state-adjoint space. For any initial state on the slow manifold, the state-adjoint and the control are given by $p_s(t, \epsilon)$ and $u_s(t, \epsilon)$, respectively. For any initial state off the slow manifold, the trajectory rapidly approaches the slow manifold in forward time according to $p_f^l(\frac{t-t_0}{\epsilon}, \epsilon)$. The left boundary layer accounts for the deviation of the actual state from the slow manifold.

Left Boundary Layer

In the left boundary layer, the state and adjoint are written in terms of the stretched time variable $\tau = \frac{t-t_0}{\epsilon}$, giving

$$\begin{aligned} x' &= \epsilon \partial H / \partial \lambda_x,\ x(0) = x_0 \\ y' &= \partial H / \partial \lambda_y,\ y(0) = y_0 \\ \lambda_x' &= -\epsilon \partial H / \partial x,\ \lambda_x(\tau_f) = 0 \\ \lambda_y' &= -\partial H / \partial y,\ \lambda_y(\tau_f) = 0. \end{aligned} \qquad (11)$$

where $'$ denotes differentiation with respect to τ. Eq. (11) represent the dynamics as viewed on the fast time-scale. The zeroth-order approximation to the solution of Eq. (11) is found by setting $\epsilon = 0$. The modified problem in the left boundary layer then evolves with the dynamics of the reduced system, given by

$$\begin{aligned} y' &= \partial H/\partial \lambda_y,\ y(0) = y_0 \\ \lambda_y' &= -\partial H/\partial y,\ \lim_{\tau\to\infty} \lambda_y(\tau) = \lambda_{y,eq}. \end{aligned} \qquad (12)$$

where x and λ_x constants. The value $\lambda_{y,eq}$ corresponds to an equilibrium point $(y_{eq}, \lambda_{y,eq})$ of Eq. (12). Furthermore, Eq. (12) corresponds to an infinite horizon regulator problem on the fast time-scale. The remainder of this paper is concerned with the solution of infinite horizon problems.

Modified Sweep Method

The modified sweep method is an indirect method for computing numerical solutions of boundary value problems [1, 6]. It is an extension of the Computational Singular Perturbation (CSP) methodology. Its application here is to problems of the form

$$\dot{p} = \begin{bmatrix} \partial H/\partial \lambda \\ -\partial H/\partial x \end{bmatrix},\ \begin{bmatrix} x(0) = x_0 \\ \lim_{t\to\infty} \lambda(t) = \lambda_{eq} \end{bmatrix} \quad (13)$$

where $x(t)$ is the state, $\lambda(t)$ is the adjoint, and

$$p = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

is the state-adjoint or *phase* vector. The initial condition for the state is given by x_0 and the final condition for the adjoint is λ_{eq} and is assumed to correspond to a saddle-type equilibrium point $((x_{eq}, \lambda_{eq}))$ of Eq. (13). The *solution* of Eq. (13) is assumed to approach (x_{eq}, λ_{eq}) asymptotically in forward time along the stable manifold of the equilibrium point. At any point p, $G(p)$ can be written as a linear combination of a set of basis vectors

$$G(p) = \sum_{i=1}^{2n} v_i(p) q_i = V(p) q \qquad (14)$$

where $V(p) = [v_1(p), \ldots, v_{2n}(p)] \in \mathbb{R}^{2n\times 2n}$, $q = [q_1, \ldots, q_{2n}]^T \in \mathbb{R}^{2n}$, and V is assumed to vary smoothly with p. The vector q is called a *rate coordinate vector* and its components q_i, $i = 1, 2, \ldots, 2n$ are called *rate coordinates*. The vector q satisfies the equation

$$q = W(p) G(p) \qquad (15)$$

where $W(p) = V^{-1}(p)$. Differentiating Eq. (15) with respect to time along a trajectory of the Hamiltonian system of Eq. (13), the rate coordinate vector obeys the differential equation

$$\dot{q} = \left[\dot{W} V + W \mathcal{J} V\right] q = Zq \qquad (16)$$

where $\mathcal{J} = \frac{\partial G}{\partial p}$ is the Jacobian matrix of $G(p)$ and $Z = Z(t) = \dot{W} V + W \mathcal{J} V$.

The matrix V is said to form a *modal basis* if at every point in the state adjoint space, the matrix Z has the block-diagonal form

$$Z = \Lambda = \begin{bmatrix} \Lambda_u(t) & 0 \\ 0 & \Lambda_s(t) \end{bmatrix} \qquad (17)$$

and the transition matrices $\Phi_s(t, 0)$ of $\Lambda_s(t)$ and $\Phi_u(t, 0)$ of $\Lambda_u(t)$ approach zero in forward and backward time, respectively. With $Z = \Lambda$ as in Eq. (17), the matrices V and W and the vector q are given the special notations $V = A$, $W = B$, and $q = h$ to identify them as a modal basis. With Λ ordered as in Eq. (17), the matrices A and B have the ordering $A = [A_u\ A_s]$ and $B = \begin{bmatrix} B_u \\ B_s \end{bmatrix}$, respectively, and the vector h has the ordering $h = \begin{bmatrix} h_u \\ h_s \end{bmatrix}$. A non-modal basis will be ordered in the same manner. Consequently, the corresponding non-modal quantities will be $V = [V_u\ V_s]$, $W = \begin{bmatrix} W_u \\ W_s \end{bmatrix}$, and $q = \begin{bmatrix} q_u \\ q_s \end{bmatrix}$. The subscripts "$s$" and "$u$" are used to denote the *approximate* stable and unstable modes, respectively.

In general it is not possible to find modal basis vectors, so non-modal basis vectors must be used. As a result, Z is not block-diagonal and the rate coordinates are coupled. Through iteration, the coupling between the approximate stable and unstable rate coordinates can be computed. Using the value of $q_u(t)$ as an input the $3n$ first-order differential equations

$$\begin{aligned} \dot{p} &= V_u q_u + V_s q_s,\ p(0) = p_0 \\ \dot{q}_s &= \left[\dot{W}_s V + W_s J V\right] \begin{bmatrix} q_u \\ q_s \end{bmatrix},\ q_s(0) = q_{s0} \end{aligned} \qquad (18)$$

are integrated forward in time. The boundary conditions for $\lambda(0)$ and $q_s(0)$ are computed as follows. First, the equation

$$W_u \cdot G(p_0) = q_u(0) \qquad (19)$$

is solved for $\lambda(0)$ which gives a value for p_0. Then, the equation

$$q_s(0) = W_s \cdot G(p_0) \tag{20}$$

is solved for $q_s(0) = q_{s0}$. In backward time, the value of $q_s(t)$ is used as an input to the system of $3n$ first-order differential equations

$$\begin{aligned} \dot{p} &= V_u q_u + V_s q_s,\ p(\tau) = p_{eq} \\ \dot{q}_u &= \left[\dot{W}_u V + W_u J V\right] \begin{bmatrix} q_u \\ q_s \end{bmatrix},\ q_u(\tau) = 0 \end{aligned} \tag{21}$$

where "τ" represents a sufficiently large time so that the dynamics have essentially come to equilibrium. The boundary conditions $x(t_f)$ and $q_u(t_f)$ are computed as follows. First the equation

$$W_s \cdot G(p(t_f)) = q_s(t_f) \tag{22}$$

is solved for $x(t_f)$. Then, the equation

$$q_u(t_f) = W_u \cdot G(p(t_f)) \tag{23}$$

is solved for $q_u(t_f)$. The value of $q_u(t)$ from Eq. (21) is then used on the next forward sweep. For a detailed description of the modified sweep method, the reader is referred to [1].

Examples

The modified sweep method will now be applied to two problems. The first problem is a regulation problem of a mass on a spring. The second problem concerns a three state formulation of the motion of an aircraft flying in a vertical plane.

Example 1: Spring-Mass System

Consider the dynamical system of a mass connected to a spring driven by a forcing function

$$m\ddot{x} + g(x) = f(t),\ x(0) = x_0,\ \dot{x}(0) = \dot{x}_0 \tag{24}$$

where m is mass, x is the position of the spring relative to the zero force point of the spring, $f(t)$ is the force applied to the spring, and $g(x)$ is the force in the spring. Setting $x_1 = x$ and $x_2 = \dot{x}$, the system of Eq. (24) has the form

$$\begin{bmatrix} \dot{x}_1 = x_2,\ x_1(0) = x_{10} \\ \dot{x}_2 = h(x) + u(t),\ x_2(0) = x_{20} \end{bmatrix} \tag{25}$$

where $h(x) = -g(x)/m$ and $u(t) = f(t)/m$. For this example, let the spring force be given by $g(x) = k_1 x_1^3 + k_2 x_1$ where k_1 and k_2 are positive constants. Assume that it is desired to drive the system from the initial condition (x_{10}, x_{20}) while minimizing the cost function

$$J = \frac{1}{2}\int_0^\infty (q_1 x_1^2 + q_2 x_2^2 + r u^2) dt. \tag{26}$$

The Hamiltonian is given by

$$H = \frac{1}{2}(q_1 x_1^2 + q_2 x_2^2 + r u^2) + \lambda_1 x_2 + \lambda_2 (h(x) + u). \tag{27}$$

The first-order necessary conditions for optimality lead to the following infinite horizon HTPBVP:

$$\begin{aligned} \dot{x}_1 &= x_2,\ x_1(0) = x_{10} \\ \dot{x}_2 &= -\tfrac{k_1 x_1^3 + k_2 x_1}{m} - \lambda_2/r,\ x_2(0) = x_{20} \\ \dot{\lambda}_1 &= -q_1 x_1 + \lambda_2 \tfrac{k_2 + 3k_1 x_1^2}{m},\ \lim_{t\to\infty} \lambda_1(t) = 0 \\ \dot{\lambda}_2 &= -q_2 x_2 - \lambda_1,\ \lim_{t\to\infty} \lambda_2(t) = 0 \end{aligned} \tag{28}$$

For this problem, the basis vectors are taken to be the eigenvectors of the Jacobian of Eq. (28) evaluated at the equilibrium point $x_{1,eq} = x_{2,eq} = \lambda_{1,eq} = \lambda_{2,eq} = 0$. Results from applying the modified sweep method for the initial condition $x_1(0) = 1$, $x_2(0) = 0$ and the values $k_1 = k_2 = q_1 = q_2 = r = 1$ are shown in Figures 1-4 alongside the exact solution on the interval $t \in [0, 10]$sec. The first important feature to observe is that each of the sweeps the values of x_1, x_2, λ_1, and λ_2 approach constants as $t \to 10$; the explosive behavior has been removed. It can be seen that by the 7^{th} sweep, the solution obtained by the modified sweep method matches closely with the exact solution.

Example 2: Aircraft Flight

The motion of an aircraft flying in a vertical plane can be described by the equations

$$\begin{aligned} \dot{E} &= (T - D)\tfrac{V}{W} \\ \epsilon \dot{h} &= V \sin\gamma \\ \epsilon \dot{\gamma} &= \tfrac{g}{V}(n - \cos\gamma) \end{aligned} \tag{29}$$

where E is the energy altitude, h is the altitude in [m], γ is the flight path angle in [rad], V is the velocity in [m/s] and g is the acceleration due to gravity in [m/s^2]. The parameter ϵ is introduced to identify the time-scale separation and to facilitate the application of the singular perturbation technique. The load factor, n, is the control for this problem. The aerodynamic model used here is that of [10]. The problem considered is to move the aircraft from an initial state $[E_0, h_0, \gamma_0]$ to a final state $[E_f, h_f, \gamma_f]$ in minimum time where the initial state is near a point where $M = V/a \approx 1$ and the final state is at a point with a significantly higher energy and altitude from the initial state. The cost function is

$$J = \int_0^{t_f} dt \tag{30}$$

and the Hamiltonian is given by

$$H = 1 + \lambda_E (T - D)\frac{V}{W} + \lambda_h V \sin\gamma + \lambda_\gamma (n - \cos\gamma)\frac{g}{V} \tag{31}$$

The 1^{st}-order necessary conditions lead to the adjoint equations

$$\dot{\lambda}_E = -\partial H/\partial E$$
$$\epsilon\dot{\lambda}_h = -\partial H/\partial h \tag{32}$$
$$\epsilon\dot{\lambda}_\gamma = -\partial H/\partial \gamma$$

The optimal control is found from

$$n^* = \arg\min_u H \tag{33}$$

The solution to this problem is known to have three segments: an initial rapid dive through the transonic regime, a slow energy climb, and a rapid ascent to meet the terminal condition. Here we focus on the rapid dive. On the fast time-scale, the dynamics of Eq. (29) and Eq. (32) form the HTPBVP

$$\begin{aligned} E' &= \epsilon(T - D)\tfrac{V}{W} \\ h' &= V \sin\gamma \\ \gamma' &= \tfrac{g}{V}(n^* - \cos\gamma) \\ \lambda_E' &= -\epsilon\partial H/\partial E \\ \lambda_h' &= -\partial H/\partial h \\ \lambda_\gamma' &= -\partial H/\partial\gamma \end{aligned} \tag{34}$$

where H now refers to the Hamiltonian evaluated at $n = n^*$. The zeroth-order approximation is found by setting $\epsilon = 0$ which leads to the reduced problem

$$\begin{aligned} h' &= V\sin(\gamma),\ h(0) = h_0 \\ \gamma' &= \tfrac{g}{V}(n^* - \cos(\gamma)),\ \gamma(0) = \gamma_0 \\ \lambda_h' &= -\partial H/\partial h,\ \lim_{\tau\to\infty}\lambda_h = \lambda_{h,eq} \\ \lambda_\gamma' &= -\partial H/\partial\gamma,\ \lim_{\tau\to\infty}\lambda_\gamma = \lambda_{\gamma,eq} \end{aligned} \tag{35}$$

where $E = E_0$ and $\lambda_E = \lambda_{E,0}$ are constants.

The modified sweep method is demonstrated for a time interval of 125 sec with $E_0 = 14700$ m and $\lambda_{E,0} = -0.066935753$. The basis vectors are taken to be the eigenvectors of the Jacobian of Eq. (35) at the equilibrium point and the initial value of $q_u(t)$ is taken to be zero. For this problem the initial states are $h(0) = 10668$ m and $\gamma(0) = 0.235$ rad. The final adjoints that correspond to equilibrium are $\lambda_h(t_f) = 0$ and $\lambda_\gamma(t_f) = -1.355$. Figures 5 - 7 show the first three forward sweeps for the states h and γ and the adjoints λ_h and λ_γ alongside the exact solution. It can be seen that with each successive forward sweep that the final values of the adjoints $\lambda_h(t_f)$ and $\lambda_\gamma(t_f)$ get closer to their known final values. This trend continues with more sweeps, but later iterations are not shown because the changes are too small to be noticeable on the plots. It can be seen that no unstable behavior is present in any of the sweeps, including the first. Furthermore, with each sweep the final values of the adjoint approach their known final values, although the convergence is very slow. A topic of continued research is the slow convergence of this problem.

Comments on the Solutions

One of the difficulties with indirect methods for the solution of HTPBVP's is that errors made in the unknown initial adjoints tend to be amplified rather than attenuated. Since unstable dynamics are present, there often is difficulty in computing a numerical solution. However, it can be seen from both examples that trajectories for all iterations show no unstable behavior. In particular, it is seen that even for the first few iterants that errors made in the adjoints leads to forward integrations that approach a constant.

Conclusions

The singular perturbation approach was applied to separate the slow and fast dynamics for a two time-scale optimal control problem. The zeroth-order approximation to the problem in the initial boundary layer is an infinite horizon problem on the fast time-scale. The proposed method of solution for an infinite horizon regulator problem is a new indirect method called *the modified sweep method.* To demonstrate the effectiveness of modified sweeping, two examples were studied. Tisphe method possesses two important features. First, none of the sweeps exhibit any unstable behavior. Second, converged solutions are known to satisfy the original equations and boundary conditions.

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